

ANOTHER SIMPLE PROOF OF AN IDENTITY CONJECTURED BY LACASSE

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Abstract. In this note, using the derangement polynomials and their umbral representation, we give another simple proof of an identity conjectured by Lacasse in the study of the PAC-Bayesian machine learning theory.

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1. INTRODUCTION

In his thesis, Lacasse introduced the functions $\xi(n)$ and $\xi_2(n)$ in the study of the PAC-Bayesian machine learning theory, where

$$\begin{aligned}\xi(n) &= \sum_{k=0}^n \binom{n}{k} \left(\frac{k}{n}\right)^k \left(1 - \frac{k}{n}\right)^{n-k}, \\ \xi_2(n) &= \sum_{k=0}^n \sum_{j=0}^{n-k} \binom{n}{k} \binom{n-k}{j} \left(\frac{k}{n}\right)^k \left(\frac{j}{n}\right)^j \left(1 - \frac{k}{n} - \frac{j}{n}\right)^{n-k-j}.\end{aligned}$$

Based on numerical verification, Lacasse presented the following conjecture.

Conjecture 1.1. *For any integer $n \geq 1$, there holds*

$$(1.1) \quad \xi_2(n) = \xi(n) + n.$$

Recently, by applying the Hurwitz identity on multivariate Abel polynomials, Younsi [7] gave an algebraic proof of this conjecture. Later, using a decomposition of triply rooted trees into three doubly rooted trees, Chen, Peng and Yang [1] gave it a nice combinatorial interpretation.

In this note, using the derangement polynomials and their umbral representation, we provide another simple proof of (1.1).

2. THE DERANGEMENT POLYNOMIALS AND THE PROOF OF (1.1)

Recall that the derangement polynomials $\{\mathcal{D}_n(\lambda)\}_{n \geq 0}$ are defined by

$$(2.1) \quad \mathcal{D}_n(\lambda) = \sum_{k=0}^n \binom{n}{k} D_k \lambda^{n-k},$$

where $\mathcal{D}_n(1) = n!$ and $\mathcal{D}_n(0) = D_n$ is the n -th derangement number, counting permutations on $[n] = \{1, 2, \dots, n\}$ with no fixed points. The derangement polynomials $\mathcal{D}_n(\lambda)$, also called λ -factorials of n , have been considerably investigated by Eriksen, Freij and Wästlund [2], Sun

and Zhuang [6]. The derangement polynomials $\mathcal{D}_n(\lambda)$ have the following basic property [2] and an Abel-type formula [6],

$$(2.2) \quad \mathcal{D}_n(\lambda + \mu) = \sum_{k=0}^n \binom{n}{k} \mathcal{D}_k(\lambda) \mu^{n-k},$$

$$(2.3) \quad \mathcal{D}_n(\lambda + \mu) = \sum_{k=0}^n \binom{n}{k} (\lambda + k)^k (\mu - k - 1)^{n-k},$$

and obey the recursive relation [6],

$$(2.4) \quad \sum_{k=0}^n \binom{n}{k} \mathcal{D}_k(\lambda) \mathcal{D}_{n-k}(\mu + 1) = (\lambda + \mu - 1)^{n+1} + (n - \lambda - \mu + 2) \mathcal{D}_n(\lambda + \mu).$$

Denote by \mathbf{D} the umbral operator defined by $\mathbf{D}^n = D_n$ for $n \geq 0$ (See [3, 4, 5] for more information on the umbral calculus), then by (2.1) $\mathcal{D}_n(\lambda)$ can be represented as

$$\mathcal{D}_n(\lambda) = (\mathbf{D} + \lambda)^n.$$

Setting $\lambda = 0, \mu = n + 1$ in (2.4), we have

$$\begin{aligned} n^{n+1} + \mathcal{D}_n(n + 1) &= \sum_{k=0}^n \binom{n}{k} \mathcal{D}_k(0) \mathcal{D}_{n-k}(n + 2) \\ &= \sum_{k=0}^n \binom{n}{k} \mathcal{D}_k(0) \mu^{n-k} \big|_{\mu=(\mathbf{D}+n+2)} \\ &= \mathcal{D}_n(\mu) \big|_{\mu=(\mathbf{D}+n+2)} \quad \text{by (2.2)} \\ &= \sum_{k=0}^n \binom{n}{k} k^k (\mu - k - 1)^{n-k} \big|_{\mu=(\mathbf{D}+n+2)} \quad \text{by (2.3)} \\ &= \sum_{k=0}^n \binom{n}{k} k^k \mathcal{D}_{n-k}(n - k + 1), \end{aligned}$$

which proves (1.1), if one notices that $\xi(n)$ and $\xi_2(n)$, by (2.3), can be rewritten as

$$\xi(n) = \frac{1}{n^n} \mathcal{D}_n(n + 1) \quad \text{and} \quad \xi_2(n) = \frac{1}{n^n} \sum_{k=0}^n \binom{n}{k} k^k \mathcal{D}_{n-k}(n - k + 1).$$

Remark 2.1. By the nontrivial property of \mathbf{D} [6],

$$(\mathbf{D} + \lambda)(\mathbf{D} + \lambda + n + 1)^n = (n + \lambda)^{n+1},$$

one can get another expression for $\xi_2(n)$,

$$\begin{aligned} \xi_2(n) &= \xi(n) + n = \frac{1}{n^n} (\mathcal{D}_n(n + 1) + n^{n+1}) \\ &= \frac{1}{n^n} ((\mathbf{D} + n + 1)^n + \mathbf{D}(\mathbf{D} + n + 1)^n) \\ &= \frac{1}{n^n} ((\mathbf{D} + 1)(\mathbf{D} + n + 1)^n) = \frac{1}{n^n} \sum_{k=0}^n \binom{n}{k} (\mathbf{D} + 1)^{k+1} n^{n-k} \\ &= \frac{1}{n^n} \sum_{k=0}^n \binom{n}{k} \mathcal{D}_{k+1}(1) n^{n-k} = \frac{1}{n^n} \sum_{k=0}^n \binom{n}{k} (k + 1)! n^{n-k}. \end{aligned}$$

This expression has been obtained by Younsi using an identity of Hurwitz on multivariate Abel polynomials and plays a critical role in his proof.

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